Vectors

Scalar Quantities: Quantities such as length, area, volume, temperature, and time, which have magnitude (size), but no direction.

Vector Quantities: Quantities that involve both a magnitude and direction, such as velocity, acceleration, and force. These quantities can be represented by **directed line segments** called **vectors**.

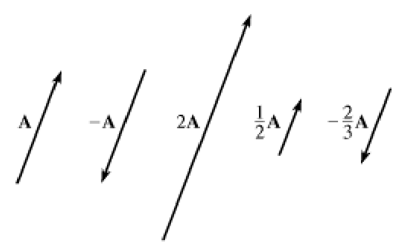
The length of a vector represents the **magnitude** of the vector quantity. The **direction** is indicated by the position of the vector and the arrowhead at one end.

Notation: $\overrightarrow{AB}$ is used to name a vector with **initial point** *A* and **terminal point** *B*. Vectors may also be denoted by bold letters. $\overrightarrow{AB}$ can also be written as $\mathbf{AB}$. If the initial and terminal points are not specified, vectors can be named by a single uppercase or lowercase letter (eg. $\vec{b}$, $\vec{B}$, $\mathbf{b}$, or $\mathbf{B}$.) The magnitude of vector $\mathbf{A}$ is written $|\mathbf{A}|$.

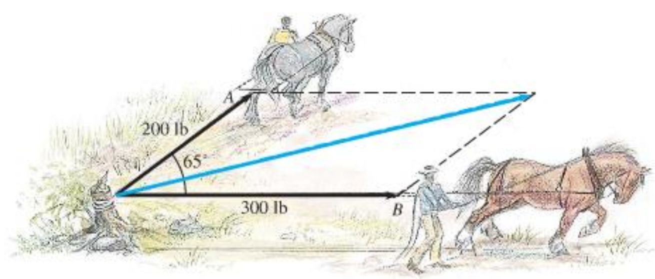
Equal Vectors: Vectors with the same magnitude and direction. They do not have to be in the same place.

Zero Vector: A vector with no magnitude and no direction. It is denoted by $\mathbf{0}$.

Scalar Multiplication: For any scalar *k* and vector **A,** $k\mathbf{A}$ is a vector with magnitude $|k|$ times the magnitude of **A**. If $k > 0$, then the direction of $k\mathbf{A}$ is the same as the direction of **A**. If $k < 0$, the direction of $k\mathbf{A}$ is opposite to the direction of **A**. If $k = 0$, then $k\mathbf{A} = \mathbf{0}$.

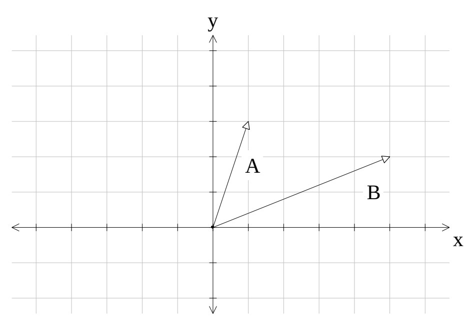


Suppose two draft horses are pulling on a tree stump with forces of 200 pounds and 300 pounds, with an angle of 65° between the forces. If **A** and **B** had the same direction, then there would be a total force of 500 pounds acting on the stump, but the total force is less because of the angle between the forces. By the **parallelogram law**, the force acting along the diagonal of the parallelogram, with a magnitude equal to the length of the diagonal, has the same effect on the stump as the two forces **A** and **B**. The force **A + B** acting along the diagonal is called the **sum** or **resultant** of **A** and **B**.

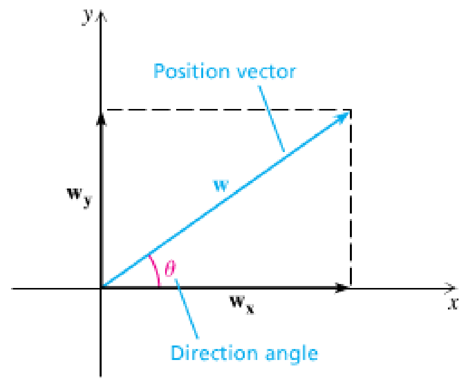


Vector Addition: To find the resultant or sum **A + B** of any vectors **A** and **B**, position **B** (without changing its magnitude or direction) so that the initial point of **B** coincides with the terminal point of **A**. The vector that begins at the initial point of **A** and ends at the terminal point of **B** is the vector **A + B**. For every vector **A**, there is a vector **– A**, with the same magnitude as **A**, but the opposite direction. For any two vectors **A** and **B**, **A – B = A +** (**– B**).

Example: Sketch the vectors **A + B** and **A – B**.



Any nonzero vector **w** is the sum of a **horizontal component**, $\mathbf{w_x}$, and a **vertical component**, $\mathbf{w_y}$. If a vector **w** is placed in a rectangular coordinate system so that its initial point is the origin, then **w** is called a **position vector**. The angle θ formed by the positive x-axis and a position vector is the **direction angle** for the position vector.



If the vector **w** has magnitude r, direction angle θ, horizontal component $\mathbf{w_x}$, and vertical component $\mathbf{w_y}$, then we get

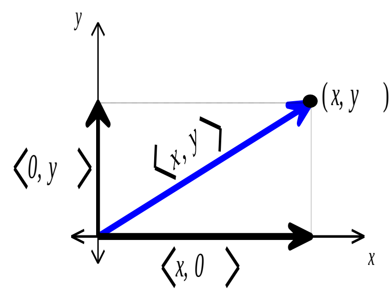
$\cos\theta = \dfrac{|\mathbf{w_x}|}{r}$ and $\sin\theta = \dfrac{|\mathbf{w_y}|}{r}$ or $|\mathbf{w_x}| = |r\cos\theta|$ and $|\mathbf{w_y}| = |r\sin\theta|$.

Examples: Find the magnitude of the horizontal and vertical components for each vector **v** with the given magnitude and direction angle θ. Round to the nearest tenth.

a) $|\mathbf{v}| = 5.6,\ \theta = 22°$

b) $|\mathbf{v}| = 445,\ \theta = 211.1°$

Component Form: The notation $\langle x, y\rangle$ is used to define a position vector with terminal point (x, y). This is called component form because the horizontal component is $\langle x, 0\rangle$ and its vertical component is $\langle 0, y\rangle$.



The magnitude of the vector $\mathbf{v} = \langle x, y\rangle$ is $|\mathbf{v}| = r = \sqrt{x^2 + y^2}$. To find the direction angle, use $\sin\theta = \dfrac{y}{r}$, $\cos\theta = \dfrac{x}{r}$, and $\tan\theta = \dfrac{y}{x}$.

If a vector has magnitude r and direction angle θ, its component form is $\langle r\cos\theta,\ r\sin\theta\rangle$.

Examples: Find the magnitude and direction angle of each vector.

a) $\mathbf{v} = \langle 2, -6\rangle$

b) $\mathbf{v} = \langle -3, 2\rangle$

Examples: Find the component form for each vector **v** with the given magnitude and direction angle θ.

a) $|\mathbf{v}| = 12,\ \theta = 45°$

b) $|\mathbf{v}| = 50,\ \theta = 120°$

Vector Arithmetic: If $\mathbf{A} = \langle a_1, a_2 \rangle$, $\mathbf{B} = \langle b_1, b_2 \rangle$, and k is a scalar, then

1. $k\mathbf{A} = \langle ka_1, ka_2 \rangle$ **Scalar Product**
2. $\mathbf{A} + \mathbf{B} = \langle a_1 + b_1, a_2 + b_2 \rangle$ **Vector Sum**
3. $\mathbf{A} - \mathbf{B} = \langle a_1 - b_1, a_2 - b_2 \rangle$ **Vector Difference**
4. $\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2$ **Dot Product**

Examples: Let $\mathbf{w} = \langle -1, -3 \rangle$ and $\mathbf{v} = \langle 3, -4 \rangle$. Perform the operations indicated.

a) $\mathbf{w} - \mathbf{v}$ b) $-8\mathbf{v}$ c) $3\mathbf{w} + 4\mathbf{v}$ d) $\mathbf{w} \cdot \mathbf{v}$

The Angle Between Two Vectors:

If **A** and **B** are nonzero vectors and α is the smallest positive angle between them, then $\cos\alpha = \dfrac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}$.

Examples: Find the smallest positive angle between the following vectors:

a) $\langle 1, 3 \rangle$ and $\langle 5, 2 \rangle$ b) $\langle -1, 5 \rangle$ and $\langle 2, 7 \rangle$

Parallel Vectors: The vectors **A** and **B** are parallel if and only if $\mathbf{A} = k\mathbf{B}$ for a nonzero scalar k.
Perpendicular Vectors: The vectors **A** and **B** are perpendicular if and only if $\mathbf{A} \cdot \mathbf{B} = 0$.

Examples: Determine whether each pair of vectors is parallel, perpendicular, or neither.

a) $\langle -2, 3 \rangle$ and $\langle 6, 4 \rangle$ b) $\langle 2, -5 \rangle$ and $\langle -4, 10 \rangle$ c) $\langle 2, 6 \rangle$ and $\langle 6, 2 \rangle$

The vectors $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ are called **unit vectors** because each has magnitude one. For any vector $\langle a_1, a_2 \rangle$, we have $\langle a_1, a_2 \rangle = a_1 \langle 1, 0 \rangle + a_2 \langle 0, 1 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$. The form $a_1\mathbf{i} + a_2\mathbf{j}$ is called a **linear combination** of the vectors **i** and **j**.

Examples: Write each vector as a linear combination of the unit vectors **i** and **j**.

a) $\mathbf{A} = \langle 2, 3 \rangle$ b) $\mathbf{B} = \langle -1, 7 \rangle$ c) $\mathbf{C} = \langle 0, -9 \rangle$